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Regularity conditions and the simplicity of prime factor rings

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Abstract

R denotes a ring with unity and $\mathbf{N}_r(R)$ its nil radical. R is said to satisfy conditions: (1) $\mathrm{pm}(\mathbf{N}_r)$ if every prime ideal containing $\mathbf{N}_r(R)$ is maximal; (2) WCI if whenever $a, e \in R$ such that $e = e^2$, $eR + \mathbf{N}_r(R) = RaR + \mathbf{N}_r(R)$, and $xe - ex \in \mathbf{N}_r(R)$ for any $x \in R$, then there exists a positive integer m such that $a^m(1-e) \in a^m\mathbf{N}_r(R)$. For example, if R is right weakly π-regular or every idempotent of R is central, then R satisfies WCI. Many authors have considered the equivalence of condition pm (i.e., every prime ideal is maximal) with various generalizations of von Neumann regularity over certain classes of rings including commutative, PI, right duo, and reduced. In the context of weakly π-regular rings, we prove the following two theorems which unify and extend nontrivially many of the previously known results.

Theorem I. *Let R be a ring with $\mathbf{N}_r(R)$ completely semiprime. Then the following conditions are equivalent*: (1) *R is right weakly π-regular*; (2) *$R/\mathbf{N}_r(R)$ is right weakly π-regular and R satisfies WCI*; (3) *$R/\mathbf{N}_r(R)$ is biregular and R satisfies WCI*; (4) *for each $\chi \in R$ there exists a positive integer m such that $R = R\chi^m R + r(\chi^m)$.*

Theorem II. *Let R be a ring such that $\mathbf{N}_r(R)$ is completely semiprime and R satisfies WCI. Then the following conditions are equivalent*: (1) *R is right weakly π-regular*; (2) *$R/\mathbf{N}_r(R)$ is right weakly π-regular*; (3) *$R/\mathbf{N}_r(R)$ is biregular*; (4) *R satisfies $\mathrm{pm}(\mathbf{N}_r)$*; (5) *if P is a prime ideal such that $\mathbf{N}_r(R/P) = 0$, then R/P is a simple domain*; (6) *for each prime ideal of R such that $\mathbf{N}_r(R) \subseteq P$, then $P = \overline{O}_P$.*

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Throughout this paper all rings are associative, but may not necessarily have a unity. All prime ideals are assumed to be proper. A ring is said to satisfy pm if *every prime ideal is maximal.* The relationship between biregularity and condition pm has been studied in [23]. Other connections between various generalizations of von Neumann regularity and condition pm have been investigated by many authors [2, 3, 6, 7–9, 12, 14, 15, 21, 24, and 25]. The earliest result of this type seems to be by Cohen [9, Theorem 1]. The survey paper [4] gives an overview of the research in this area.

For the case of commutative rings, the first clearly established equivalence between condition pm and a generalization of von Neumann regularity seems to have been made by Storrer [21] in the following result: If R is a commutative ring with unity, then the following statements are equivalent: (1) R is π-regular; (2) $R/\mathbf{P}(R)$ is von Neumann regular ($\mathbf{P}(R)$ is the prime radical of R); (3) R satisfies pm. Fisher and Snider generalized this result to P.I. rings [12, Theorem 2.3], while Chandran extended Storrer's result to duo rings [8, Theorem 3]. Next, Hirano generalized Chandran's result to right duo rings [15, Corollary 1]. Furthermore, Hirano's result was extended to bounded weakly right duo rings by Yao [24, Theorem 3].

Recently, on the other hand, for the case of reduced rings (i.e., rings without nonzero nilpotent elements) Beidar and Wisbauer [2], Belluce [3], Birkenmeier, Kim and Park [6], and Camillo and Xiao [7] showed the following generalization of Storrer's result: If R is a reduced ring with unity, then the following statements are equivalent: (1) R is biregular; (2) R is weakly regular; (3) R is right weakly π-regular; (4) R satisfies condition pm; (5) every prime factor ring of R is a simple domain.

Two general questions seem to underlie the above-mentioned papers.

(1) What is the connection between the various generalizations of regularity and the condition that all prime ideals (or some specific subclass, such as the completely prime ideals) are primitive ideals or maximal ideals?

(2) For a ring R, when can some type of regularity condition on R/I be "lifted" to some type of regularity on condition on R, where I is a certain type of ideal (e.g., I is some type of radical ideal of R)? (In general, regularity conditions pass to homomorphic images).

In this paper we unify and extend many of the results of the previously mentioned papers by investigating the above two questions in the context of right weakly π-regular rings. The class of right weakly π-regular rings not only includes the class of π-regular rings and the class of right fully idempotent rings, but it is also related to the class of I-rings [16, p. 210] since every nonnil ideal contains a nonzero idempotent right ideal. Applications and examples are provided to illustrate and delimit the theory developed herein.

Throughout this paper R denotes a ring. For $X \subseteq R$, $\langle X \rangle_R$ and $r(X)$ denote the ideal generated by X (when R has unity we use RXR) and the right annihilator of X, respectively. We use $\mathbf{P}(R), \mathbf{N}_r(R), \mathbf{J}(R), \mathbf{BM}(R)$, and $\mathbf{N}(R)$ to represent the prime radical of R, the nil radical of R, the Jacobson radical of R, the Brown–McCoy radical of R, and the set of all nilpotent elements of R, respectively.

Recall that an ideal P of R is *completely prime* (*completely semiprime*) if $ab \in P$ implies $a \in P$ or $b \in P$ (if $a^2 \in P$ implies $a \in P$) for $a, b \in R$. R is called a *2-primal* ring if $\mathbf{P}(R) = \mathbf{N}(R)$. Historically, some of the earliest results known to us about 2-primal rings (although not so called at the time) and prime ideals were due to Shin [19]. For example, he showed that a ring R is 2-primal if and only if every minimal prime ideal of R is completely prime. Hirano [15] considered the 2-primal condition in the context of strongly π-regular rings. He used the term *N-ring* for what we call a 2-primal ring. Also the 2-primal condition was taken up independently by Sun [22], where he introduced a condition called *weakly symmetric*, which is equivalent to the 2-primal condition for rings. Sun [22] showed that if R is weakly symmetric, then each minimal prime ideal of R is a completely prime ideal, and that the ring of n-by-n upper triangular matrices over R inherits the weakly symmetric condition. The name *2-primal rings* originally and independently came from the context of left near rings by Birkenmeier et al. in [5].

Observe R is 2-primal if and only if $\mathbf{P}(R) = \mathbf{N}_r(R) = \mathbf{N}(R)$ if and only if $\mathbf{P}(R)$ is completely semiprime. Also $\mathbf{N}_r(R) = \mathbf{N}(R)$ if and only if $\mathbf{N}_r(R)$ is completely semiprime.

We use $\mathrm{pm}(\rho)$ to denote the condition that *every prime ideal of R containing $\rho(R)$ is a maximal ideal of R*, where ρ symbolizes a radical (if $\rho = \mathbf{P}$, we just use pm). Finally, R is said to be *reduced* if $\mathbf{N}(R) = 0$.

1. Preliminaries

In this section we develop some of the basic properties of the $\mathrm{pm}(\rho)$ condition for various radicals ρ. Also we show a connection between the right weak π-regularity of $R/\rho(R)$ and the simplicity of prime factors R/P, where P is a prime ideal containing $\rho(R)$. Examples are provided which delimit the relationship between the pm condition and various regularity conditions.

Routine arguments show that the class of pm rings includes biregular rings and rings with d.c.c. on right ideals. Furthermore, a full matrix ring over a pm ring with unity is again a pm ring. Note that although every nonzero prime ideal is maximal in the ring $\mathbb{Z}$ of integers, $\mathbb{Z}$ does not satisfy condition pm, since zero is a prime ideal.

Lemma 1.1. *Let R be a ring, I an ideal of R and ρ a hereditary radical. If Q is a prime ideal of I with $\rho(I) \subseteq Q$, then there exists a prime ideal P of R such that $Q = I \cap P$ and $\rho(R) \subseteq P$.*

Proof. By Andrunakievich's Lemma, Q is an ideal of R. Hence,

$$P = \{v \in R \mid Iv \subseteq Q\}$$

is an ideal of R. We *claim* P is a prime ideal of R. Let $x, y \notin P$. Assume that $\langle Ix\rangle_R\langle Iy\rangle_R \subseteq P$. Then $\langle Ix\rangle_I\langle Iy\rangle_I \subseteq P$. So $I\langle Ix\rangle_I\langle Iy\rangle_I \subseteq Q$. Since $Q \neq I$, either $\langle Ix\rangle_I \subseteq Q$

or $\langle Iy\rangle_I \subseteq Q$. Assume $\langle Ix\rangle_I \subseteq Q$. Then $Ix \subseteq Q$, so $x \in P$, a contradiction! Hence, P is a prime ideal of R. Since, $I\rho(R) \subseteq I \cap \rho(R) = \rho(I)$, then $\rho(R) \subseteq P$. □

Proposition 1.2. *Let ρ be a hereditary radical. Then the class of rings satisfying* $\mathrm{pm}(\rho)$ *is closed under*:
(1) *homomorphic images*;
(2) *ideals*;
(3) *direct sums.*

Proof. The proof of parts (1) and (3) are routine. For part (2), let I be an ideal of R and Q be a prime ideal of I with $\rho(I) \subseteq Q$. Then by Lemma 1.1, there exists a prime ideal P of R such that $Q = I \cap P$ and $\rho(R) \subseteq P$. Then $I/Q \cong (I+P)/P = R/P$ which is a simple ring since P is maximal in R. Thus, I has $\mathrm{pm}(\rho)$. □

Note Lemma 1.1 and Proposition 1.2(2) are adaptations of the proof of Theorem 2.6 of [12].

Definition 1.3. (1) A ring R is called *right* (*left*) *weakly regular* if $I^2 = I$ for each right (left) ideal I of R, equivalently $x \in x\langle x\rangle_R$ ($x \in \langle x\rangle_R x$) for each $x \in R$. R is *weakly regular* if it is both left and right weakly regular [17]. Note that right (left) weakly regular rings are also called *right* (*left*) *fully idempotent*.

(2) A ring R is called (*strongly*) *π-regular* if for every $x \in R$ there exists a positive integer $n = n(x)$, depending on x, such that ($x^n \in x^{n+1}R$) $x^n \in x^nRx^n$. Strong π-regularity is left-right symmetric [11].

(3) A ring R is called *right* (*left*) *weakly π-regular* if for every $x \in R$ there exists a positive integer $n = n(x)$, depending on x, such that $x^n \in x^n\langle x^n\rangle_R$ ($x \in \langle x^n\rangle_R x^n$). R is *weakly π-regular* if it is both left and right weakly π-regular [13]. Biregular rings (e.g., simple rings), right V-rings, and π-regular rings are right weakly π-regular rings. Observe that if R is right weakly π-regular, then every nonnil right ideal contains a nonzero idempotent right ideal.

The following examples provide some limitations as well as some motivation for the relation between generalized von Neumann regularity conditions and condition pm.

Example 1.4 [12, Example 1]. Let R consist of all sequences of 2-by-2 matrices over a field which are eventually strictly upper triangular. This ring is semiprime and it satisfies pm, but R is not von Neumann regular.

Example 1.5. The ring of endomorphisms of a countably infinite-dimensional vector space provides a ring which is von Neumann regular but does not satisfy pm.

Example 1.6 [3, p. 1865]. Let R be the ring of all sequences of 2-by-2 matrices over a field which are eventually diagonal. Then R is von Neumann regular and satisfies pm, but R is not biregular.

Example 1.7. This example is a generalization of [6, Example 12], which shows that the weak regularity of $R/\mathbf{P}(R)$ cannot necessarily be lifted to R: Let W be a simple domain with unity which is not a division ring (e.g., a Weyl algebra over a field of characteristic 0) and let R be the 2-by-2 upper triangular matrix ring over W. Clearly R is 2-primal and $R/\mathbf{P}(R) \cong W \oplus W$ is a biregular (hence weakly regular) ring. We *claim* that R is neither left nor right weakly π-regular. To see this let xW be a nonzero proper right ideal of W and assume that R is right weakly π-regular. Then there exists n such that

$$\begin{pmatrix} x & 1 \\ 0 & 0 \end{pmatrix}^n \in \begin{pmatrix} x & 1 \\ 0 & 0 \end{pmatrix}^n R \begin{pmatrix} x & 1 \\ 0 & 0 \end{pmatrix}^n R.$$

Observe

$$\begin{pmatrix} x & 1 \\ 0 & 0 \end{pmatrix}^n = \begin{pmatrix} x^n & x^{n-1} \\ 0 & 0 \end{pmatrix}$$

and

$$\begin{pmatrix} x^n & x^{n-1} \\ 0 & 0 \end{pmatrix} \begin{pmatrix} a & b \\ 0 & c \end{pmatrix} \begin{pmatrix} x^n & x^{n-1} \\ 0 & 0 \end{pmatrix} \begin{pmatrix} u & v \\ 0 & w \end{pmatrix} \in \begin{pmatrix} x^n W x^n W & x^n W x^{n-1} W \\ 0 & 0 \end{pmatrix}$$

for $a, b, c, u, v, w \in W$ and $n > 1$. Therefore,

$$\begin{pmatrix} x & 1 \\ 0 & 0 \end{pmatrix}^n \in \begin{pmatrix} x^n W x^n W & x^n W x^{n-1} W \\ 0 & 0 \end{pmatrix}.$$

Hence $x^{n-1} = x^n \alpha$, where $\alpha \in W x^{n-1} W$. So $x^{n-1}(1 - x\alpha) = 0$. Then $1 \in xW$, a contradiction! Therefore, R is not right weakly π-regular. Similarly, R is not left weakly π-regular.

This example also shows that the class of weakly π-regular rings is neither a radical class nor a semisimple class since it is not closed under extensions.

Example 1.8. The ring in Example 1.4 and any simple domain with unity are semiprime and satisfy $\mathfrak{pm}$. Therefore if a ring R is semiprime and it satisfies $\mathfrak{pm}$, it is not necessarily von Neumann regular. However, [6, Corollary 9] gives some support that semiprime rings with $\mathfrak{pm}$ may be one-sided weakly π-regular. But the following example, which is a generalization of [6, Example 13], shows that this, in general, is not the case: Let W be a simple domain with unity which is not a division ring. Let R be the ring of all sequences of 2-by-2 matrices over W which are eventually constant upper triangular. A routine argument shows that R is semiprime. The proof that R satisfies $\mathfrak{pm}$ is the same as that given in [6, Example 13]. Note that R is not 2-primal.

To see that R is neither right nor left weakly π-regular, let $s \in R$ such that the first component is

$$\begin{pmatrix} x & 1 \\ 0 & 0 \end{pmatrix},$$

where xW is a nonzero proper right ideal of W and zero in all other components of s. Then as in Example 1.7, $s^m \notin s^m R s^m R$, for any positive integer m. Similarly, R is not weakly π-regular.

Proposition 1.9. *$\rho(R)$ is completely semiprime if and only if every prime ideal of R, which is minimal among prime ideals of R containing $\rho(R)$, is completely prime (where $\rho(R) = \mathbf{P}(R)$, $\mathbf{N}_r(R)$, $\mathbf{J}(R)$, or $\mathbf{BM}(R)$).*

Proof. ($\Rightarrow$) Let P be a prime ideal of R which is minimal among prime ideals of R containing $\rho(R)$. Then $P/\rho(R)$ is a minimal prime ideal of $R/\rho(R)$. Since $R/\rho(R)$ is reduced, then $P/\rho(R)$ is completely prime (see [18, p. 202] or observe that $R/\rho(R)$ is 2-primal). Hence, P is completely prime.

($\Leftarrow$) Let B be the intersection of all prime ideals which are minimal among prime ideals of R containing $\rho(R)$. Let D be the intersection of all prime ideals of R containing $\rho(R)$ (The case for $\rho(R) = \mathbf{P}(R)$ is routine).

Case 1: $\rho(R) = \mathbf{N}_r(R)$. Recall that a strongly prime ideal of R contains $\mathbf{N}_r(R)$ (see [18, p. 200]). Then $\mathbf{N}_r(R) \subseteq B = D \subseteq \bigcap\{\text{strongly prime ideals of } R\} = \mathbf{N}_r(R)$ (see [18, p. 201]).

Case 2: $\rho(R) = \mathbf{J}(R)$. Recall that a primitive ideal of R is a prime ideal of R which contains $\mathbf{J}(R)$. Then $\mathbf{J}(R) \subseteq B = D \subseteq \bigcap\{\text{primitive ideals of } R\} = \mathbf{J}(R)$.

Case 3: $\rho(R) = \mathbf{BM}(R)$. Let M be an ideal of R such that R/M is a simple ring with unity. Then M is a prime ideal of R containing $\mathbf{BM}(R)$. Hence $\mathbf{BM}(R) \subseteq B = D \subseteq \bigcap\{M \text{ is an ideal of } R \mid R/M \text{ is a simple ring with unity}\} = \mathbf{BM}(R)$.

Thus in all three cases $B = \rho(R)$. Since, B is completely semiprime, then so is $\rho(R)$. □

Proposition 1.10. *If $\rho(R)$ is a completely semiprime ideal of R and $R/\rho(R)$ is right weakly π-regular, then R/P is a simple domain with unity for every prime ideal P of R with $\rho(R) \subseteq P$ (where $\rho(R) = \mathbf{P}(R), \mathbf{N}_r(R), \mathbf{J}(R)$, or $\mathbf{BM}(R)$).*

Proof. Let P be a prime ideal of R such that $\rho(R) \subseteq P$. Then there exists a prime ideal X of R which is minimal among prime ideals of R containing $\rho(R)$ and $X \subseteq P$. By Proposition 1.9, X is completely prime. Let $\overline{R} = R/X$. Then $\overline{R}$ is a right weakly π-regular domain. Let v be a nonzero element of $\overline{R}$. There exists a positive integer k such that $v^k = v^k y$, where $y \in \overline{R} v^k \overline{R}$. Hence, $v^k(y^2 - y) = 0$. Since $\overline{R}$ is a domain, then y is a unity for $\overline{R}$. So $\overline{R}$ is a simple ring with unity and $X = P$. □

2. Main results

Example 1.7 shows that the 2-primal condition is not strong enough to "lift" weak π-regularity from $R/\mathbf{P}(R)$ to R. In this section we provide the missing "lifting" condition. In our main results, Theorems 2.6 and 2.8, we prove that if R satisfies the WCI condition (defined below) and $\mathbf{N}_r(R) = \mathbf{N}(R)$, then the following conditions are equivalent: (1) R is right weakly π-regular; (2) $R/\mathbf{N}_r(R)$ is right weakly π-regular; (3) R satisfies condition $\mathfrak{pm}(\mathbf{N}_r)$.

Applications of these results show that they extend previously known results on reduced rings and on weakly right duo rings.

Throughout this section R denotes a ring with unity.

Definition 2.1. We say a ring R satisfies *WCI* (*weakly commuting idempotents*) if whenever $a, e \in R$ such that $e = e^2$, $eR + \mathbf{N}_r(R) = RaR + \mathbf{N}_r(R)$, and $xe - ex \in \mathbf{N}_r(R)$ for any $x \in R$, then there exists a positive integer m such that $a^m(1-e) \in a^m\mathbf{N}_r(R)$.

Observe that in Definition 2.1 we can replace the statement "$a^m(1-e) \in a^m\mathbf{N}_r(R)$" with the seemingly weaker "$a^m(1-e) \in a^mX$, where $X = \{x \in R \mid x(1-e) \in \mathbf{N}_r(R) + r(a^m) + RaR\}$". However, the definition so obtained remains equivalent to the original definition. Since Definition 2.1 is of crucial importance in the sequel, we develop several basic properties in the next lemma.

Lemma 2.2. (1) *Let $a, e \in R$ such that $e = e^2$, $eR + \mathbf{N}_r(R) = RaR + \mathbf{N}_r(R)$, and $xe - ex \in \mathbf{N}_r(R)$ for any $x \in R$. Then $RaR(1-e) \subseteq \mathbf{N}_r(R)$.*

(2) *If R is right weakly π-regular, then R satisfies WCI.*

(3) *If every idempotent element of R is central, then R satisfies WCI.*

Proof. (1) Routine arguments show that $RaR(1-e) \subseteq \mathbf{N}_r(R)$.

(2) Let $a, e \in R$ as described in the hypothesis of part (1). Since R is weakly π-regular there exists a positive integer m such that $a^m(1-e) \in a^mRa^mR(1-e) \subseteq a^mRaR(1-e) \subseteq a^m\mathbf{N}_r(R)$. Thus, R satisfies WCI.

(3) Let $a, e \in R$ as described in the hypothesis of part (1). Observe $a(1-e) \in \mathbf{N}_r(R)$. Then there exists a positive integer m such that $0 = (a(1-e))^m$. But $1-e$ is central so $a^m(1-e) = 0 = a^m0$. Thus, $a^m(1-e) \in a^m\mathbf{N}_r(R)$. Therefore, R satisfies WCI. □

The ring of 2×2 matrices or 2×2 upper triangular matrices over a division ring provides a counterexample to the converse of Lemma 2.2(3). However the next result, which is due to V. Camillo and M. May (via private communication) illustrates the connection between the WCI condition and the condition that all idempotents are central.

Lemma 2.3. *R has every idempotent central if and only if whenever ac is nilpotent then there exists a positive integer m such that $a^mc = 0$ for $a, c \in R$ with $c = c^2$.*

Proof. Assume every idempotent is central and $c = c^2$. Then $(ac)^m = a^mc$. So, if $(ac)^m = 0$ then $a^mc = 0$. Conversely, assume whenever ac is nilpotent then there exists a positive integer m such that $a^mc = 0$ for $a, c \in R$ with $c = c^2$. For purpose of contradiction, suppose that not every idempotent is central. Then there exists $e = e^2$ such that $(1-e)Re \neq 0$. Let $b \in R$ such that $(1-e)be \neq 0$. Let $a = (1-e)be + (1-e)$. So $a = a^2 \neq 0$. Now $(ae)^2 = 0$, but $a^me = ae = (1-e)be \neq 0$, a contradiction. Thus every idempotent is central. □

Lemma 2.4. *Let I be a nil ideal. Then R/I is biregular if and only if for $a \in R$ there exists $e = e^2 \in R$ such that $RaR = eR \oplus X$, where $X = RaR \cap (1-e)R \subseteq I$ and $er - re \in I$ for all $r \in R$.*

Proof. Let $a \in R$. Since R/I is biregular and idempotents lift modulo a nil ideal, there exists $e = e^2 \in R$ such that $er - re \in I$ for all $r \in R$ and $RaR + I = ReR + I = eR + I$. Then $e = \alpha + x$, where $\alpha \in RaR$ and $x^n = 0$ for some n. Hence, $e = e^n = (\alpha + x)^n \in RaR$. Therefore, $RaR = eR \oplus X$, where $X = RaR \cap (1-e)R$ and $X \subseteq I$. The converse follows immediately. □

Observe that if R/I is biregular then $\mathbf{P}(R) \subseteq I$.

Lemma 2.5. *Let R be a ring such that $\rho(R)$ is completely semiprime, where $\rho(R) = \mathbf{P}(R)$, $\mathbf{N}_r(R)$, or $\mathbf{J}(R)$. If $R/\rho(R)$ is biregular and $a \in R$, then there exists $c \in R$ such that $c - c^2 \in \rho(R)$, $xc - cx \in \rho(R)$ for all $x \in R$, $RaR + \rho(R) = cR + \rho(R)$, and $R = Ra^nR + (1-c)R$ for any positive integer n.*

Proof. Since $R/\rho(R)$ is biregular, there exists $c \in R$ such that $c - c^2 \in \rho(R)$, $RaR + \rho(R) = cR + \rho(R)$, and $xc - cx \in \rho(R)$ for all $x \in R$. Let $\overline{R} = R/\rho(R)$, $\overline{a} = a + \rho(R)$, $\overline{c} = c + \rho(R)$, and $\overline{1} = 1 + \rho(R)$. Then $\overline{R} = \overline{c}\overline{R} \oplus (\overline{1} - \overline{c})\overline{R}$. Thus, $R = RaR + (1-c)R + \rho(R) = RaR + (1-c)R$, since $\rho(R)$ is superflous. Observe that no proper ideal can contain both a^n and $1-c$. Otherwise a maximal ideal M of R would contain a^n and $1-c$. Since every prime ideal of $R/\rho(R)$ is maximal, Proposition 1.9 yields that M is completely prime. Hence $a \in M$, a contradiction. Therefore, $R = Ra^nR + R(1-c)R$. So $c = \alpha + \beta$ and $c = c^2 + \gamma$ where $\alpha \in Ra^nR$, $\beta \in R(1-c)R$, and $\gamma \in \rho(R)$. Thus, $c = c\alpha + c\beta + \gamma$. Now $c\beta \in cR(1-c)R$, but $\overline{c}\overline{R}(\overline{1} - \overline{c}) \subseteq \mathbf{N}(\overline{R})$. Since $\overline{R}$ is reduced, $cR(1-c) \subseteq \rho(R)$. Therefore, $c \in Ra^nR + \rho(R)$. Consequently, $R = cR + \rho(R) + (1-c)R = Ra^nR + \rho(R) + (1-c)R = Ra^nR + (1-c)R$. □

Observe that since idempotents lift modulo a nil ideal, if $\rho(R) = \mathbf{P}(R)$ or $\mathbf{N}_r(R)$ then c (in Lemma 2.5) can be taken to be an idempotent of R. Recently, in [25] Yu introduced *right quasi-duo rings* (i.e., rings in which every maximal right ideal is an ideal). He showed that $\mathbf{N}(R) \subseteq \mathbf{J}(R)$ if R is right quasi-duo. However more can be said: if R is a right quasi-duo ring, every maximal ideal is completely prime, hence $\mathbf{J}(R)$ is completely semiprime. To see this, one need only observe that if M is a maximal ideal, then R/M is a division ring. Thus from [25, Theorem 4.4], right quasi-duo P-exchange rings have the properties indicated in Lemma 2.5. Observe in this case $\mathbf{J}(R) = \mathbf{N}_r(R)$.

Our next result shows that the WCI condition with $\mathbf{N}_r(R)$ completely semiprime will allow us to "lift" the right weakly π-regular condition from $R/\mathbf{N}_r(R)$ to R.

Theorem 2.6. *Let R be a ring with $\mathbf{N}_r(R)$ completely semiprime. Then the following conditions are equivalent:*

(1) *R is right weakly π-regular;*

(2) *$R/\mathbf{N}_r(R)$ is right weakly π-regular and R satisfies WCI;*

(3) *$R/\mathbf{N}_r(R)$ is biregular and R satisfies WCI;*
(4) *for each $a \in R$ there exists a positive integer m such that $R = Ra^mR + r(a^m)$.*

Proof. (1)$\Rightarrow$(2). This implication follows from Lemma 2.2 and the fact that right weak π-regularity is preserved under homomorphisms.

(2)$\Rightarrow$(3). Since $R/\mathbf{N}_r(R)$ is reduced, this implication is a consequence of [6, Theorem 8] and either [7, Theorem 6] or [2, Corollary 4.3].

(3)$\Rightarrow$(4). Lemma 2.5 and the comment immediately following it yield the existence of $c \in R$ such that $c^2 = c$, $xc - cx \in \mathbf{N}_r(R)$ for all $x \in R$, $RaR + \mathbf{N}_r(R) = cR + \mathbf{N}_r(R)$, and $R = Ra^nR + (1-c)R$ for any positive integer n. By condition WCI, there exists a positive integer m such that $a^m(1-c) = a^m w$ where $w \in \mathbf{N}_r(R)$. Hence $1-c-w \in r(a^m)$. So $1-c \in \mathbf{N}_r(R)+r(a^m)$. Consequently, $R = Ra^mR+(1-c)R = Ra^mR+\mathbf{N}_r(R)+r(a^m) = Ra^mR + r(a^m)$.

(4)$\Rightarrow$(1). This implication is immediate. □

In [19], to each prime ideal P, Shin associates the set $O_P = \{a \in R \mid ab = 0$ for some $b \in R \setminus P\}$.

Definition 2.7. Let P be a prime ideal of R. We use $\overline{O}_P$ to denote the set $\{a \in R \mid a^n \in O_P$ for some positive integer $n\}$.

Our next result illustrates the relationship between condition $\mathfrak{pm}(\mathbf{N}_r)$ and right weak π-regularity.

Theorem 2.8. *Let R be a ring such that $\mathbf{N}_r(R)$ is completely semiprime and R satisfies WCI. Then the following conditions are equivalent:*
(1) *R is right weakly π-regular;*
(2) *$R/\mathbf{N}_r(R)$ is right weakly π-regular;*
(3) *$R/\mathbf{N}_r(R)$ is biregular;*
(4) *R satisfies $\mathfrak{pm}(\mathbf{N}_r)$;*
(5) *if P is a prime ideal of R such that $\mathbf{N}_r(R/P) = 0$, then R/P is a simple domain;*
(6) *for each prime ideal P of R such that $\mathbf{N}_r(R) \subseteq P$, then $P = \overline{O}_P$.*

Proof. The equivalences (1)$\Leftrightarrow$(2)$\Leftrightarrow$(3) follow from Theorem 2.6. The equivalence (3)$\Leftrightarrow$(4) is a consequence of [23, Theorem 1.10].

(4)$\Rightarrow$(5). Observe that if P is a prime ideal such that $\mathbf{N}_r(R/P) = 0$, then $\mathbf{N}_r(R) \subseteq P$. Hence, R/P is a simple ring. By Proposition 1.9, R/P is a domain.

(5)$\Rightarrow$(4). Let P be a prime ideal such that $\mathbf{N}_r(R) \subseteq P$. There exists a prime ideal X which is minimal among prime ideals containing $\mathbf{N}_r(R)$ and $X \subseteq P$. By Proposition 1.9, R/X is a domain. Hence, $\mathbf{N}_r(R/X) = 0$. So X is a maximal ideal of R. Hence, P is a maximal ideal of R.

(3)$\Rightarrow$(6). Let P be a prime ideal of R with $\mathbf{N}_r(R) \subseteq P$ and $a \in P$. By Lemma 2.5, there exists $e = e^2$ such that $xe - ex \in \mathbf{N}_r(R)$ for all $x \in R$, $RaR + \mathbf{N}_r(R) =$

$eR + \mathbf{N}_r(R)$, and $R = Ra^nR + (1-e)R$ for any positive integer n. By condition WCI, there exists a positive integer m such that $a^m(1-e) = a^m w$ where $w \in \mathbf{N}_r(R)$. Hence $a^m(1-e-w)=0$. Since P is properly contained in R, $1-e \notin P$. But $w \in P$, so $1-e-w \notin P$. Thus, $P \subseteq \overline{O}_P$. To show $\overline{O}_P \subseteq P$, let $d \in \overline{O}_P$. Then there exists $b \in R \setminus P$ such that $d^k b = 0$, for some positive integer k. Since R has $\mathrm{pm}(\mathbf{N}_r)$, Proposition 1.9 yields that P is completely prime. Hence, $d \in P$. Thus, $\overline{O}_P = P$.

(6)$\Rightarrow$(4). Let P be a prime ideal of R containing $\mathbf{N}_r(R)$ and M a maximal ideal of R such that $P \subseteq M$. Since M is a prime ideal containing $\mathbf{N}_r(R)$, we have $M = \overline{O}_M \subseteq \overline{O}_P = P$. Thus, P is a maximal ideal of R. □

Note that if R is a ring in which the WCI condition is left–right symmetric (e.g., every idempotent is central) and $\mathbf{N}_r(R)$ is completely semiprime, then the conclusions of Theorem 2.8 are left–right symmetric. In particular, part (1) can be replaced by "R is weakly π-regular". An immediate consequence of Lemma 2.2(3) and Theorem 2.8 is that if R is a local ring with $\mathbf{J}(R) = \mathbf{N}_r(R)$ then R is weakly π-regular.

Also an immediate corollary of Theorem 2.8 and the fact that in a reduced ring $O_P = \overline{O}_P$ for all prime ideals P is the following result which includes many previously known results.

Corollary 2.9 [2,3,6 and 7]. *Assume that R is a reduced ring. Then the following conditions are equivalent*:

(1) *R is weakly π-regular*;

(2) *R is right weakly π-regular*;

(3) *R satisfies condition* pm;

(4) *R is biregular*;

(5) *R is weakly regular*;

(6) *R is right weakly regular*;

(7) *every prime factor ring of R is a simple domain*;

(8) *$R = RaR + r(a)$ for each $a \in R$*;

(9) *for each prime ideal P of R, $P = O_P$.*

Note that condition (9) of our Corollary 2.9 is the same as condition (3) of Theorem 6 in [7]. Hence, Theorem 2.8 generalizes parts (1)–(3) of Theorem 6 in [7]. Furthermore, if R is reduced, a routine argument shows that condition (8) of our Corollary 2.9 implies that R is a right p.p. ring and that $RaR = R$ for all a such that $r(a) = 0$ (i.e., part (4) of Theorem 6 in [7]). Since R is reduced, the sum in part (8) of Corollary 2.9 can be considered as a direct sum.

The following definition, due to Shin [19], embodies several conditions which are relevant to our study and provides a class of rings which satisfies the hypothesis of Theorem 2.8.

Definition 2.10. A ring R is called *almost symmetric* if it satisfies the following two conditions:

(SI) the right annihilator of each element is an ideal of R;

(SII) for any $a, b, c \in R$, if $a(bc)^n = 0$ for a positive integer n, then $ab^m c^m = 0$ for some positive integer m.

Recall from [24], R is called a *weakly right duo ring* if for every $a \in R$ there is a positive integer n, depending on a, such that $a^n R$ is an ideal of R.

Proposition 2.11. *If R satisfies any of the following conditions, then $\mathbf{N}_r(R)$ is completely semiprime and R satisfies WCI:*

(1) $\mathbf{N}(R) = \mathbf{N}_r(R)$ *(e.g., R is 2-primal) and all idempotents are central.*

(2) $\mathbf{N}(R) = \mathbf{N}_r(R)$ *(e.g., R is 2-primal) and R satisfies condition* (SII).

(3) *R satisfies condition* (SI).

(4) *Every nilpotent element is central.*

(5) *R is weakly right duo.*

(6) *R is right quasi-duo, $\mathbf{J}(R) = \mathbf{N}_r(R)$, and R satisfies WCI.*

Proof. A routine argument shows that $\mathbf{N}(R) = \mathbf{N}_r(R)$ if and only if $\mathbf{N}_r(R)$ is completely semiprime.

(1) This part follows from Lemma 2.2(3).

(2) Let $a, e \in R$ such that $e = e^2$ and $(ae)^n = 0$. Then $1(ae)^n = 0$. By condition (SII), $1a^m e = a^m e = 0$ for some m. The proof of this part is completed by using Lemma 2.3 and part (1).

(3) If R satisfies condition (SI), then R is 2-primal and all idempotents are central by [19, Theorem 1.5 and Lemma 2.7]. So it satisfies condition (1).

(4) If every nilpotent element is central, a routine argument shows that R satisfies condition (1).

(5) If R is weakly right duo, then R satisfies condition (1) by [24, Lemmas 2 and 4].

(6) This part follows from the comments after Lemma 2.5. □

Recall [19] that if R satisfies condition (SI), then R is 2-primal and every idempotent is central. At this point one might ask: *if R is 2-primal, has every idempotent central, and satisfies* (SII) *and* pm, *does R satisfy* (SI)? Note any local ring R, in which the maximal ideal is $\mathbf{P}(R)$, is 2-primal with every idempotent central and satisfies conditions (SII) and pm. Observe that Example 5.1(c) of [19] is a local ring in which the maximal ideal is the prime radical. But this ring does not satisfy condition (SI), furthermore all the nilpotent elements are not central.

Lemma 2.12. (1) [19, Lemma 1.2(d)]. *A ring R satisfies* (SI) *if and only if for any a, b in R, $ab = 0$ implies $aRb = 0$.*

(2) *If R satisfies condition* (SI) *and $x \in R$ such that $x^n = 0$, then $(RxR)^n = 0$.*

Proof. Part (2) follows from part (1). □

Corollary 2.13. *If R is a ring which satisfies conditions* (SI) *and* pm, *then for each $a \in R$ there exists a positive integer k (depending on a) such that $(RaR)^k = (RaR)^{k+1}$.*

Proof. By Theorem 2.8 and Proposition 2.11, $R/\mathbf{P}(R)$ is biregular. From Lemma 2.4, there exists $e = e^2$ such that $R = RaR + (1-e)R$, where $RaR = eR \oplus X$ and $X = RaR \cap (1-e)R \subseteq \mathbf{P}(R)$. Hence there is a positive integer k such that $(a(1-e))^k = 0$. Consider $(Ra)^k R = (Ra)^k(RaR+(1-e)R) = (RaR)^{k+1}+(Ra(1-e)R)^k$. By Lemma 2.12, $(Ra(1-e)R)^k = 0$. Therefore, $(RaR)^k = (RaR)^{k+1}$. □

Note from Corollary 2.13 we also have $(aR)^{k+1} = (aR)^{k+2}$ and $(Ra)^{k+1} = (Ra)^{k+2}$.

Proposition 2.14. *Let R be weakly right duo and right weakly π-regular, then R is strongly π-regular.*

Proof. Let $a \in R$. There exist positive integers m and n such that $a^nR = Ra^nR$ and $a^mR = a^mRa^mR$. Observe that $a^{2n}R = a^na^nR = a^nRa^nR = Ra^nRa^nR = Ra^na^nR = Ra^{2n}R$. An induction argument yields $a^{kn}R = Ra^{kn}R$ for any positive integer k. Also $a^{2m}R = a^m(a^mR) = a^m(a^mRa^mR) = a^{2m}Ra^mR$. Again an induction argument yields $a^{km}R = a^{km}Ra^mR$ for any positive integer k.

Now using the above observations, we have that $a^{mn}Ra^{mn}R = a^{mn}a^{mn}R = a^{2mn}R$. Also we have that $a^{mn}Ra^{mn}R = (a^{mn}Ra^mR)a^{mn}R = a^{mn}Ra^ma^{mn}R = a^{mn}Ra^{mn+m}R = (a^{mn}Ra^mR)a^{mn+m}R = \cdots = a^{mn}Ra^{mn+2m}R = \cdots = a^{mn}Ra^{mn+mn}R = a^{mn}R(a^{mn}R\ a^{mn}R) = a^{mn}R(a^{mn}Ra^{mn}Ra^{mn}R) = (a^{mn}Ra^{mn}R)(a^{mn}Ra^{mn}R) = a^{2mn}Ra^{2mn}R = a^{4mn}R \subseteq a^{2mn+1}R \subseteq a^{2mn}R$. Hence $a^{2mn}R = a^{2mn+1}R$. Therefore, R is a strongly π-regular ring. □

The following result extends Theorems 2 and 3 of [24].

Corollary 2.15. *Let R be weakly right duo. Then the following conditions are equivalent:*

(1) *R is weakly π-regular;*
(2) *R is strongly π-regular;*
(3) *R satisfies* $\mathrm{pm}(\mathbf{N}_r)$;
(4) *$R/\mathbf{N}_r(R)$ is biregular.*

Proof. The proof follows from Theorem 2.8, Propositions 2.11(5), and 2.14. □

Recall [20] that R is a *left P-exchange ring* if every projective left R-module has the exchange property. Also note that every π-regular ring is an exchange ring [20]. For our final application in this section we have:

Corollary 2.16. *Let R be a ring all of whose idempotents are central. If R is a left P-exchange ring, then R is weakly π-regular.*

Proof. This result is a consequence of Theorem 2.8, Proposition 2.11(1), and [20, Theorem 4.8]. □

Let M be a unital (R,R)-bimodule. Recall the *split-null extension* (*or trivial extension*) $S(R,M)$ of M by R is the ring formed from the Cartesian product $R \times M$ with componentwise addition and with multiplication given by $(a,m)(b,k) = (ab, ak + mb)$. Note that R embeds into $S(R,M)$ via $a \to (a,0)$.

Lemma 2.17. *If R satisfies WCI, then $S(R,M)$ satisfies WCI.*

Proof. Let S denote $S(R,M)$. Choose $(a,x),(e,v) \in S$ such that $(e,v)^2 = (e,v)$, $(e,v)S+\mathbf{N}_r(S) = S(a,x)S+\mathbf{N}_r(S)$, and $(b,y)(e,v)-(e,v)(b,y) \in \mathbf{N}_r(S)$ for any $(b,y) \in S$. Then $e = e^2$. From the embedding of R into S, $eR+\mathbf{N}_r(R) = RaR+\mathbf{N}_r(R)$ and $te-et \in \mathbf{N}_r(R)$ for any $t \in R$. Since R satisfies the WCI condition, there exists a positive integer m such that $a^m(1-e) \in a^m\mathbf{N}_r(R)$. There exists $\bar{x} \in M$ such that $(a,x)^m = (a^m,\bar{x})$. Then $(a,x)^m(1-e,-v) = (a^m(1-e), a(-v)+\bar{x}(1-e)) = (a^m(1-e),0)+(0,a(-v)+\bar{x}(1-e)) \in \mathbf{N}_r(S)$. Therefore, S satisfies WCI. □

Observe that if $S(R,M)$ is right weakly π-regular, then R is right weakly π-regular. A partial converse is contained in the next result.

Proposition 2.18. (1) *If R is 2-primal, then $S(R,M)$ is 2-primal.*

(2) *If $\mathbf{N}_r(R) = \mathbf{N}(R)$, then $\mathbf{N}_r(S(R,M)) = \mathbf{N}(S(R,M))$.*

(3) *If $\mathbf{N}_r(R) = \mathbf{N}(R)$, then $R/\mathbf{N}_r(R) \cong S(R,M)/\mathbf{N}_r(S(R,M))$.*

(4) *Let R be a ring satisfying WCI and $\mathbf{N}_r(R) = \mathbf{N}(R)$. Then R is right weakly π-regular if and only if $S(R,M)$ is right weakly π-regular.*

Proof. (1) This part follows from [5, Proposition 2.5(ii)] and the fact that if A is a subring of B then $\mathbf{P}(A) = \mathbf{P}(B) \cap A$.

(2) This part is a consequence of the fact that $\mathbf{N}(S(R,M)) = \{(a,m) \mid a \in \mathbf{N}(R)\}$.

(3) Define $h : R \to S(R,M)/\mathbf{N}_r(S(R,M))$ by $h(x) = (x,0) + \mathbf{N}_r(S(R,M))$. Then h is a ring epimorphism with $ker(h) = \mathbf{N}_r(R)$.

(4) This part follows from parts (2) and (3), Lemma 2.17, Theorem 2.8, and the observation after Lemma 2.17. □

3. Examples and constructions

In this section we provide examples which demonstrate that our results properly extend previously known results. Several examples illustrate the precision of our results by indicating that further generalization in certain directions is limited. Finally, construction methods are introduced which provide rings satisfying the hypothesis of Theorem 2.8.

The following example is a ring which is almost symmetric, and it satisfies pm; but it is not π-regular and is neither right nor left weakly regular. However, by Theorem 2.8 and Proposition 2.11 (or Proposition 2.18(4)), it is weakly π-regular.

Example 3.1. Let W be a simple domain with unity which is not a division ring, and let

$$R = \left\{ \begin{pmatrix} a & b \\ 0 & a \end{pmatrix} \middle| \, a, b \in W \right\}.$$

Observe that R is isomorphic to the split-null extension $S(W, W)$.

Claim 1. *R satisfies* (SI).

Proof. Let $A = \left(\begin{smallmatrix} a & b \\ 0 & a \end{smallmatrix}\right)$. If $a \neq 0$, then the right annihilator of A is zero. If $a = 0$, then the right annihilator of A is $\left(\begin{smallmatrix} 0 & W \\ 0 & 0 \end{smallmatrix}\right)$. In either case the right annihilator of A is an ideal of R.

Claim 2. *R satisfies* (SII), *hence R is almost symmetric.*

Proof. Observe that $\mathbf{P}(R) = \left(\begin{smallmatrix} 0 & W \\ 0 & 0 \end{smallmatrix}\right)$ is a completely prime ideal. Assume $A, B, C \in R$ such that $A(BC)^n = 0$. Then $A(BC)^n \in \mathbf{P}(R)$. Note if $A = 0$, we are finished so assume $A \neq 0$. If $A \notin \mathbf{P}(R)$, then $(BC)^n \in \mathbf{P}(R)$. Hence, $B \in \mathbf{P}(R)$ or $C \in \mathbf{P}(R)$. Thus, $AB^2C^2 = 0$. If $A \in \mathbf{P}(R)$, then $(BC)^n$ is in the right annihilator of A which is $\mathbf{P}(R)$. So again $AB^2C^2 = 0$.

Claim 3. *R satisfies* pm, *since $\mathbf{P}(R)$ is the unique maximal ideal of R.*

Claim 4. *R is not π-regular.*

Proof. Let $a \neq 0$ and $a \neq 1$. Then

$$\begin{pmatrix} a & 0 \\ 0 & a \end{pmatrix}^n \notin \begin{pmatrix} a & 0 \\ 0 & a \end{pmatrix}^n R \begin{pmatrix} a & 0 \\ 0 & a \end{pmatrix}^n,$$

for any positive integer n.

Claim 5. *R is neither right nor left weakly regular, since $\mathbf{P}(R)^2 = 0$. Since, $R \cong S(W, W)$ and W is weakly regular, this claim shows that in Proposition* 2.18 *we cannot replace the "right weakly π-regular" condition with the "right weakly regular" condition.*

Claim 6. *R is not right quasi-duo. It can be easily checked that a ring is right quasi-duo if and only if every factor ring modulo a right primitive ideal is a division* *ring. Now*

$$R\Big/\begin{pmatrix} 0 & W \\ 0 & 0 \end{pmatrix} \cong W$$

is a simple domain, but not a division ring. So R cannot be a right quasi-duo ring.

As noted before Corollary 2.16, every π-regular ring is an exchange ring. This fact and Corollary 2.16 suggest that weakly π-regular rings with idempotents central may be P-exchange rings. However, Example 3.1 and Proposition 4.3 of [25] show that this is *not* the case. However from Example 3.1 and Proposition 2.11, one can see that Theorem 2.8 extends several of the results in [15] and [24] from the π-regular condition to the more general right weakly π-regular condition.

In [12] and [25] the condition, every prime ideal of R is right primitive, is considered in the context of π-regularity. So it is natural to ask: Can the $\mathfrak{pm}(\mathbf{N}_r)$ condition be weakened to the condition, every prime ideal containing $\mathbf{N}_r(R)$ is a right primitive ideal, in Theorem 2.8? The following example provides a *negative* answer to this question.

Example 3.2. Assume that D is a division ring which is transcendental over its center. For an example of such a division ring, let F be a field with the characteristic 0 and let $W_1[F]$ be the first Weyl algebra over F. Recall that $W_1[F] = F[\mu, \lambda]$, the polynomial ring with indeterminates μ and λ such that $\lambda\mu = \mu\lambda + 1$. Then $W_1[F]$ is a simple Noetherian domain. So its classical quotient ring D is a division ring with the center F. Indeed, let $q \in Z(D)$, the center of D. Then $(W_1[F] : q) = \{w \in W_1[F] \mid wq \in W_1[F]\}$ is a nonzero ideal of $W_1[F]$. Therefore, $(W_1[F] : q) = W_1[F]$ and so $q \in W_1[F]$. Hence $q \in F$, and so $Z(D) = F$. By noting that $\lambda \in D$, D is transcendental over its center F (in fact, D is purely transcendental over F). Therefore, the polynomial ring $D[x]$ is a primitive ring (see [10, p. 62]). Now we *claim* that every prime ideal of $D[x]$ is primitive. Let P be a prime ideal of $D[x]$. If $P = 0$, then since $D[x]$ is a primitive ring, we are done. If $P \neq 0$, then $P = f(x)D[x]$ with $f(x) \in F[x]$ and $f(x)F[x]$ is a nonzero prime ideal of $F[x]$. Now note that $F[x]/f(x)F[x]$ is a finite extension field of F and D is a central simple F-algebra. Therefore, $D[x]/P \cong D \otimes_F (F[x]/f(x)F[x])$ is a simple algebra, and hence P is a maximal ideal of $D[x]$. Therefore every prime ideal of $D[x]$ is primitive. Also the ring $D[x]$ satisfies the hypothesis of Theorem 2.8. But the ring $D[x]$ is not right weakly π-regular.

Intuitively the conditions $\mathfrak{pm}$ and $\mathfrak{pm}(\mathbf{N}_r)$ seem to be very close and the fact that $\mathfrak{pm}$ implies $\mathfrak{pm}(\mathbf{N}_r)$ may lead one to conjecture that they are equivalent. However in the following example, the ring R satisfies $\mathfrak{pm}(\mathbf{N}_r)$ but *not* $\mathfrak{pm}$. Surprisingly, $\mathbf{N}_r(R)$ is completely semiprime, but R is *not* 2-primal. Also the ring R illustrates both Theorem 2.6 and Theorem 2.8.

Example 3.3. Let G be an abelian group which is the direct sum of a countably infinite number of infinite cyclic groups; and denote by $\{b(0), b(1), b(-1), \ldots, b(i), b(-i), \ldots\}$

a basis of G. Then there exists one and only one homomorphism $u(i)$ of G, for $i = 1,2,\ldots$ such that $u(i)(b(j)) = 0$ if $j \equiv 0 \pmod{2^i}$ and $u(i)(b(j)) = b(j-1)$ if $j \not\equiv 0 \pmod{2^i}$. Denote U the ring of endomorphisms of G generated by the endomorphisms $u(1), u(2), \ldots$. Now let A be the ring obtained from U by adjoining the identity map of G. Then by [1, p. 540], the ring A is semiprime and $\mathbf{N}_r(A) = U$. Since $A/U \cong \mathbb{Z}$, U is a semiprimitive ideal and thus $\mathbf{N}_r(A) = \mathbf{J}(A) = U$.

Now let $R = A \otimes_{\mathbb{Z}} \mathbb{Q}$ and $\mathbb{Q}$ the field of rationals. Then since ${}_{\mathbb{Z}}\mathbb{Q}$ is flat, we have the following exact sequence:

$$0 \to U \otimes_{\mathbb{Z}} \mathbb{Q} \to A \otimes_{\mathbb{Z}} \mathbb{Q} \to (A/U) \otimes_{\mathbb{Z}} \mathbb{Q} \to 0$$

from the exact sequence

$$0 \to U \to A \to A/U \to 0.$$

Therefore, $(A/U) \otimes_{\mathbb{Z}} \mathbb{Q} \cong (A \otimes_{\mathbb{Z}} \mathbb{Q})/(U \otimes_{\mathbb{Z}} \mathbb{Q})$ and so $\mathbb{Q} \cong \mathbb{Z} \otimes_{\mathbb{Z}} \mathbb{Q} \cong (A \otimes_{\mathbb{Z}} \mathbb{Q})/(U \otimes_{\mathbb{Z}} \mathbb{Q})$. Thus, $\mathbf{J}(R) = U \otimes_{\mathbb{Z}} \mathbb{Q}$ and hence the ring R is a local ring with the maximal ideal $U \otimes_{\mathbb{Z}} \mathbb{Q}$. Furthermore, note $\mathbf{N}_r(A) = U$ and $\mathbf{J}(R) = U \otimes_{\mathbb{Z}} \mathbb{Q} = \mathbf{N}_r(A) \otimes_{\mathbb{Z}} \mathbb{Q} \subseteq \mathbf{N}_r(R)$ and so $\mathbf{J}(R) = \mathbf{N}_r(R) = U \otimes_{\mathbb{Z}} \mathbb{Q}$. Thus, $\mathbf{N}_r(R)$ is completely semiprime.

For our *claim* that R is semiprime, first we show that $a \otimes 1 = 0$ in R with $a \in A$ implies $a = 0$. Note that $\{b(i) \otimes 1 \mid i = 0, 1, -1, 2, -2, \ldots\}$ is a basis of the vector space $G \otimes_{\mathbb{Z}} \mathbb{Q}$ over $\mathbb{Q}$. Since the map $g : A \times \mathbb{Q} \to \mathrm{End}_{\mathbb{Q}}(G \otimes_{\mathbb{Z}} \mathbb{Q})$ defined by $g((a,q))(b(i) \otimes 1) = a(b(i)) \otimes q$ with $a \in A$ and $q \in \mathbb{Q}$ is $\mathbb{Z}$-bilinear, there exists a map $\theta : A \otimes_{\mathbb{Z}} \mathbb{Q} \to \mathrm{End}_{\mathbb{Q}}(G \otimes_{\mathbb{Z}} \mathbb{Q})$ defined by $\theta(a \otimes q)(b(i) \otimes 1) = a(b(i)) \otimes q$.

For $a \in A$, assume that $a \otimes 1 = 0$. Then $\theta(a \otimes 1) = 0$ and so $a(b(i)) \otimes 1 = 0$ for all i. Let

$$a(b(i)) = m_1 b(i_1) + m_2 b(i_2) + \cdots + m_k b(i_k)$$

with $m_1, m_2, \ldots, m_k, i_1, i_2, \ldots, i_k \in \mathbb{Z}$. Then $0 = a(b(i)) \otimes 1 = m_1(b(i_1) \otimes 1) + \cdots + m_k(b(i_k) \otimes 1)$. But since $\{b(i) \otimes 1 \mid i = 0, 1, -1, 2, -2, \ldots\}$ is a basis of the vector space $G \otimes_{\mathbb{Z}} \mathbb{Q}$ over $\mathbb{Q}$, we have that $m_1 = m_2 = \cdots = m_k = 0$. Therefore, $a(b(i)) = 0$ for all i, and so $a = 0$.

Now let $\alpha \in R$ such that $\alpha R \alpha = 0$. Then $\alpha = a \otimes (1/n)$ for some $a \in A$ and some nonzero integer n. Therefore, we have that $aAa \otimes_{\mathbb{Z}} \mathbb{Q} = 0$. Particularly, $aba \otimes 1 = 0$ for every $b \in A$. So it follows that $aba = 0$ for all $b \in A$ by the above argument, and hence $aAa = 0$. Since A is semiprime, $a = 0$ and so $\alpha = 0$. Consequently, the ring R is semiprime.

Note that since the ring R is local and $\mathbf{J}(R) = \mathbf{N}_r(R)$, it is right quasi-duo and weakly right duo, but not bounded weakly right duo (see [24, Lemma 6]). By Corollary 2.15, it is strongly π-regular.

Let S be a ring (not necessarily with unity) and C a commutative ring with unity ($\neq 0$) such that S is a C-algebra. Let $(S; C)$ denote the Dorroh extension of S via C, that is the C-algebra with unity defined on $S \times C$ with the following operations:

$$(s_1, c_1) + (s_2, c_2) = (s_1 + s_2, c_1 + c_2) \tag{i}$$

$$c(s_1, c_1) = (cs_1, cc_1) \tag{ii}$$

$$(s_1, c_1) \cdot (s_2, c_2) = (s_1 s_2 + c_1 s_2 + c_2 s_1, c_1 c_2) \tag{iii}$$

for $s_1, s_2 \in S$ and $c, c_1, c_2 \in C$.

The following result allows us to construct rings satisfying the hypothesis of Theorem 2.8.

Proposition 3.4. *Let S be a C-algebra which is a ρ-radical ring (where ρ is the prime, nil, or Jacobson radical). Take $R = (S; C)$. Then:*

(i) *$\rho(R)$ is completely semiprime;*

(ii) *if ρ is either the prime or the nil radical, then R satisfies the WCI condition.*

Proof. The proof is routine. □

Example 3.5. Let S be a nil radical ring.

(1) If $R = (S; \mathbb{Z})$, then R satisfies the hypothesis of Theorem 2.8. However, R is *not* right weakly π-regular, since $(R/\mathbf{N}_r(R)) \cong \mathbb{Z}$ is not biregular.

(2) If the characteristic of S is n and $R = (S; \mathbb{Z}_n)$, then R satisfies the hypothesis of Theorem 2.8 and R is right weakly π-regular, since $(R/\mathbf{N}_r(R)) \cong \mathbb{Z}_n$ is strongly π-regular (hence right weakly π-regular).

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